International Journal of Theoretical Physics, Vol. 20, No. 3, 1981

The Superposition of the States and the Logic Approach to Quantum Mechanics

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Received May 28, 1980

An axiomatic approach to quantum mechanics is proposed in terms of a "logic" scheme satisfying a suitable set of axioms. In this context the notion of pure, maximal, and characteristic state as well as the superposition relation and the superposition principle for the states are studied. The role the superposition relation plays in the reversible and in the irreversible dynamics is investigated and its connection with the tensor product is studied. Throughout the paper, the W^*-algebra model, which satisfies our axioms, is used to exemplify results and properties of the general scheme.

1. INTRODUCTION

The so-called logic approach to classical and quantum mechanics has its beginning in 1936 with a paper by Birkhoff and von Neumann. In that paper the authors study the connection between the set of the experimental observations of the physical system Σ and subsets of the "phase space" of Σ which is assumed to be the set of the pure state of Σ.

In the classical case the connection there studied is such that to every Lebesgue measurable region of the phase space one can associate the experimental proposition (test, yes–no experiment, or simply proposition) which establishes whether the position and momentum coordinates of the system are in that region or not.

If one identifies regions whose difference has Lebesgue measure zero, one is then led to represent the experimental propositions in terms of a distributive orthocomplemented lattice (Boolean algebra) of classes of measurable subsets of the phase space of Σ.

The tests associated in the above-mentioned way to Σ ensure a complete description of all the observables of the system. Indeed, every

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classical observable is represented by a measurable function on the phase space and hence it can be analyzed in terms of the yes–no experiments corresponding to the inverse images of the values it takes on the real line.

In the quantum case the connection is such that one is led to identify the yes–no experiments on Σ with the closed subspaces $L(H)$ of the hilbert space H of Σ. The yes–no experiments are indeed observables of Σ which assign to the presence or absence of a certain property relative to Σ the value 1 or 0, respectively.

Hence they can be identified with the orthogonal projections on H or equivalently with the closed subspaces of H (von Neumann, 1955). Also here the yes–no experiments on Σ give a complete description of the observable of Σ since, owing to the spectral theorem of the self-adjoint operators, every observable can be analyzed in terms of a suitable collection of projections and hence of tests of Σ.

There is a natural physical interpretation of the inclusion ($\leqslant$) of the subspaces in $L(H)$: $a, b \in L(H)$ and $a \leqslant b$ means that the answer yes for the tests of a implies the answer yes for the tests of b. Moreover for every test $a \in L(H)$, the Hilbertian orthogonal complement $a^{\perp}$ of a represents the test obtained from a by interchanging its outcomes. Given $a, b \in L(H)$ the subspace $a \cap b$ (denoted $a \wedge b$) of H gives the test which answers yes when both a and b answer yes, while the closed linear span $a \vee b$ of a and b in H defines a new test through $a \vee b = (a^{\perp} \wedge b^{\perp})^{\perp}$.

With the operations $\leqslant, \wedge, \vee, \perp$ the set $L(H)$ of the yes–no experiments on Σ becomes a lattice which is orthomodular, namely, it has a weak property of distributivity (a Boolean algebra is an orthomodular lattice, but the converse does not hold) (Birkhoff and von Neumann, 1936).

Birkhoff and von Neumann associate then axiomatically to every physical system Σ the set of the yes–no experiments on Σ whose structure is assumed to be that of an orthomodular lattice (*logic* of Σ) and whose operations are interpreted and motivated as in the previous classical and quantum examples.

From the point of view of the logics the difference between classical and quantum systems is given by the "propositional calculus," which consists in the classical case by the operations of a Boolean algebra and in the quantum case by the more general calculus of the operations of a nondistributive orthomodular lattice.

One of the problems is now of finding under what conditions the initially mentioned classical and quantum models can be obtained as special cases of the assumed abstract scheme.

According to standard results in the theory of the distributive lattices, a Boolean logic can be represented in terms of subsets of a set (see Birkhoff and von Neumann, 1936, and references therein).

On the other hand the mathematical problem of representing a nondistributive logic has not had a simple history. In their paper Birkhoff and von Neumann were able to determine the class of modular irreducible logics of finite length >3 as the class of the projective geometries on number fields admitting involutorial antiisomorphisms with definite diagonal Hermitian form.

The problem was successively attacked by Jauch and Piron in their axiomatic approach to quantum mechanics (Piron, 1964, 1976; Jauch, 1968; Jauch and Piron, 1963, 1969). Piron was able to generalize the results obtained by Birkhoff and von Neumann characterizing the logics which can be represented in terms of irreducible Hilbertian logics, namely, in terms of the closed subspaces of a Hilbert space over the real, complex, or quanternionic numbers (Piron, 1964) (see also Section 3 of this paper).

From a physical point of view Piron's result provides a different and direct motivation about the Hilbert space which is associated to the physical system in the conventional quantum mechanics.

We now have to consider also the states of the system since the propositions alone are not sufficient to get physical previsions. If one wants to know the probability of the outcome yes of a yes–no experiment performed on the physical system, one has to specify the preparing procedures (states) by which it has been prepared.

In classical statistical mechanics this is obtained representing the states by means of the probability measures on a Boolean logic of the phase space of the system (Mackey, 1963; Gudder, 1970).

The natural extension of this point of view to quantum statistical mechanics is that of assuming that the states be represented by σ-additive measures (with total mass 1) from the (nondistributive) logic of the system to the interval $[0, 1]$.

This works well in the Hilbert model since, by a fundamental theorem due to Gleason (Gleason, 1957) the σ-additive measures (with total mass 1) on the Hilbertian logic $L(H)$ (H a separable Hilbert space) can be identified with the statistical operators on H.

Once the classical and the irreducible quantum systems have been characterized the problem arises of studying logics and states of more general physical systems (there exist indeed quantum physical systems which are not irreducible, namely, possessing superselection rules).

In doing a physical theory, one typically associates to the physical system the propositions [or even the observables or the effects (Ludwig, 1974)], the states and a function which gives the probability of the outcome yes for a proposition (or the probability distribution for an observable or for an effect) when the system is in a given state together with a suitable set of axioms which must be satisfied by the theory. Examples of such physical theories are, beside the ordinary quantum mechanics, the scheme proposed by Mackey (1963), Jauch (1968), Varadarajan (1968), Piron (1976), and Ludwig (1974) (a review of mathematical structures of quantum mechanics is the content of Beltrametti and Cassinelli, 1976).

Among the different physical theories the so-called logic approach to quantum mechanics consists of those theories in which the propositions are assumed as the fundamental observables (even if not necessarily the primitive entities). A widely studied problem in the context of the logic approach to quantum mechanics is the one connected with the superposition principle. It is well known that this is a point where some distinguishing properties between classical and quantum physical systems can be made very transparent. Dirac (1947) based his formulation of quantum mechanics directly on this principle. By means of it, it is possible to obtain new pure states by linearly combining and normalizing to 1 representative vectors of rays (pure states) of the Hilbert space of the physical system.

The language of lattice theory is particularly convenient for a mathematically precise formulation of the superposition of the states and of the quantum (or classical) superposition principle.

A formulation in terms of atomic propositions has been given by Jauch (1968), which shows that the quantum superposition principle is in fact a consequence or the non-Boolean structure of the logic of the physical system.

It has also been shown that if a superposition principle holds, then the logic is a complete atomic lattice (Gudder, 1970) and that, if the latter is Boolean, the only nontrivial superpositions of the states are the statistical mixtures of state (Gudder, 1970; Varadarajan, 1968). The formulation of the superposition principle given by Jauch implies the irreducibility of the logic, namely, the absence of superselection rules for the physical system. Such a result has been obtained also by using Varadarajan's superposition of states together with a quantum superposition principle for pure states (Varadarajan, 1968; Pulmannovà, 1976). However, an atomic non-Boolean logic, even if reducible, admits of quantum superpositions of states (Berzi and Zecca, 1974).

The superposition principle has also been formulated as a modification of the formula which gives the statistical mixture for pure states (Delyannis, 1976) and, in the context of the algebraic quantum theory, in a way similar to the one proposed by Jauch (Chen, 1973). There exists also a formulation in terms of transition probabilities in the context of Mackey's scheme (Cantoni, 1975, 1976).

Another problem, even if less studied, of the logic approach to quantum mechanics is that of providing a dynamical picture for the physical system directly in the language of its abstract formulation. There exist indeed definitions of the reversible dynamical evolution of the system in terms of one parameter group of convex automorphisms of the set of states (Schrödinger-type dynamics) (Mackey, 1963; Varadarajan, 1968) or in terms of orthoisomorphisms of the logic (Heisenberg-type dynamics) (Jauch, 1968; Píron, 1976). However, the results are generally obtained once the scheme has been specialized to the Hilbertian model.

The same can be said for the formulation of the irreversible time evolution of the physical system (Piron, 1976).

As far as the author knows, the only study of a reversible dynamics developed at the level of an abstract proposition-state structure is the one proposed by Gorini and Zecca (1975). The invariance under time translation of the superposition relation for the states is there employed to have physically equivalent Schrödinger and Heisenberg pictures.

Other problems connected with the logic approach to quantum mechanics will be mentioned in Section 4.

The object of this paper is to provide a logic scheme in which the problems mentioned can be studied possibly as a basis for further developments.

We associate to the physical system Σ the propositions and the states of Σ together with a set of physical and mathematical assumptions (such a theory will *be called proposition-state structure*). The mutual conceptual dependence of states and propositions is made evident by a suitable set of axioms, without assuming the propositions to be fundamental entities. In this sense we follow the point of view by Pool (1968). (See also Gallone and Zecca 1973.)

The mathematical assumptions on the logic are such that the physical system to which the logic refers is allowed to have any kind (even continuous) of superselection rules. For this reason, some results of the paper are an extension of results obtained in Varadarajan's framework, where, however, the superselection rules are assumed to be at most discrete (Varadarajan, 1968; Pulmannovà, 1976). For a discussion of the algebraic representation of the superselection rules see Cirelli, Gallone, and Gubbay (1975) and Cirelli and Gallone, 1973).

Motivated by the statistical interpretation, the states are assumed to be a convex subset of the set of the additive measures on the logic. They are also assumed to be strongly order determining on the logic and to have a further property of "normality" analogous to the property of the normal states of a W^* algebra (Sakai, 1971).

In the context of those assumptions, the concepts of characteristic, pure, and maximal state are discussed. These concepts, which coincide in the standard Hilbert model (Gallone and Zecca 1973) and which have been studied also in Berzi and Zecca (1974), are further investigated in some interesting classes of proposition-state structures (the problem remains open, however, whether they coincide in a general proposition-state structure).

The superposition relation for the states is formulated in a way equivalent to the one of Varadarajan. Both a classical and a quantum superposition principle are formulated by means of the superposition relation restricted to the pure states. The physical systems in which a classical or a quantum superposition principle holds are recovered to be the classical and the purely quantum systems, respectively.

The reversible dynamics for the physical system proposed in Gorini and Zecca (1975) is reformulated in a compact way and applied to the case in which a discrete decomposition of the states in terms of pure states is possible. It is shown that the projections and the normal states of a W^* algebra satisfy the axioms which define a proposition-state structure.

Almost in every section of the paper the W^*-algebra model is used to exemplify definitions and results obtained at the level of a general proposition-state structure. In particular it is shown that the pure, characteristic, and maximal states coincide in this model and that the problem of the reversible dynamics can be completely solved by using results of Kadison.

Also the C^*-algebra model indirectly fits our axioms as a consequence of the GNS construction, and in that connection some physical situations are discussed.

Finally the scheme is specialized to get the standard irreducible Hilbert model. With such a representation it is easily seen that the definition of superposition of states we use is a natural extension to the statistical operators of Dirac's superposition of pure states.

We then prove that, in the Hilbert model, the superposition relation of the statistical operators is preserved also under the most general (linear irreversible) dynamical evolution for the physical system (hence in particular by a dynamics on the statistical operators governed by an homogeneous generalized master equation).

As a last result we show that the superposition relation for the statistical operators is preserved also under the coupling of physical systems when the context of the Hilbert model is assumed.

2. THE PROPOSITION-STATE STRUCTURE

2.1. Mathematical and Physical Assumptions. We start with the definition of the scheme that will be used through the paper. The physical interpretation will immediately follow. We use Maeda and Maeda's book (1970) as the reference for the lattice theory. Notations and some useful definitions and results concerning logics are collected in the Appendix.

Definition 2.1. A proposition-state structure (pss) is a pair (L, S), where L is a logic, namely, a complete orthomodular lattice (whose elements will be called propositions) and S is a family of maps (states) from L to the interval $[0,1]$ such that, by setting $S_1(a)=\{s\in S\colon s(a)=1\}$ $(a\in L)$, the following holds:

A1: $a, b\in L\Rightarrow(a\leqslant b\Leftrightarrow S_1(a)\subset S_1(b))$ (S is strongly order determining on L);

A2: $S_1(\bigwedge_\alpha a_\alpha)=\bigcap_\alpha S_1(a_\alpha)\ \forall\{a_\alpha\}\subset L$ ("normality" property of the states);

A3: $s\in S$; $a, b\in L$, $a\perp b\Rightarrow s(a\vee b)=s(a)+s(b)$; $s(\mathbb{1})=1$ (the states are additive measures on L with total mass 1);

A4: S is convex, namely, $s_1, s_2\in S$, $\alpha\in[0,1]\Rightarrow\alpha s_1+(1-\alpha)s_2\in S$;

A5: $d\in C(L)$, $s\in S$, $s(d)\neq 0\Rightarrow s_d\in S$, s_d being defined by $s_d(x)=s(x\wedge d)/s(d)\ \forall x\in L$ [$C(L)$ denotes the center of L which is a distributive sublogic of L].

With our assumptions the logic L is a generalized proposition system in the sense of Piron (1964). The name "proposition-state structure" has been introduced in analogy with the name "event-state structure" introduced by Pool (1968).

We note that the additivity of the states implies, by induction, the property

$$s\left(\bigvee_{i=1}^{n} a_i\right)=\sum_{i=1}^{n} s(a_i)\qquad (s\in S)$$

for every finite family $\{a_i\}$ of mutually orthogonal elements of L. Indeed from $a_i\leqslant(a_k^\perp)$ $(i\neq k)$ there follows $a_i\leqslant(\bigvee_{k\neq i}a_k)^\perp\ \forall i$.

From A4 and by induction we have also that if $\{s_i\}\subset S$ is a finite family then

$$\sum_{i=1}^{n}\alpha_i s_i\in S\qquad\forall\{\alpha_i\}\subset[0,1]\qquad\text{with }\sum_{i=1}^{n}\alpha_i=1$$

Moreover, S being convex, the following definition makes sense.

Definition 2.2. A *pure state* of a *pss* (L, S) is an element $s\in S$ such that $s=\alpha s_1+(1-\alpha)s_2$ $(s_1, s_2\in S,\ \alpha\in[0,1])\Rightarrow s=s_1=s_2$.

The set of the pure states will be denoted by S_p.

The physical interpretation of a proposition-state structure is the following. To every physical system Σ we associate a *pss* (L, S). The propositions of L are interpreted to represent classes of equivalent observation procedures of Σ having only two possible outcomes, say "yes" and "no" (tests), for instance an excited or nonexcited counter or a filter which lets a certain kind of particles pass (a particle passes or does not pass).

The states of S are interpreted to represent the preparing procedures for Σ, namely, instructions for an apparatus to produce samples of Σ. By considering a sample of Σ which has been prepared according to an $s\in S$, and by observing it with an $a\in L$ the number $s(a)$ gives the probability of the answer yes for a test of a.

From the axiom A1 we have that $a=b$ if and only if $S_1(a)=S_1(b)$ and hence a proposition is completely determined by the states which are completely true on it.

If now $\{a_\alpha\}$ is any family of elements of L, from A1 we have $S_1(\bigwedge_\alpha a_\alpha)\subset\bigcap_\alpha S_1(a_\alpha)$, that is, the answer yes for a test of $\bigwedge_\alpha a_\alpha$ implies the answer yes for any test of any a_α. By requiring A2 we assume that the converse holds, namely, that the tests of $\bigwedge_\alpha a_\alpha$ are defined by giving the answer yes with certainty if and only if every test of every a_α gives the answer yes with certainty. Even if A2 seems to be a strong request on the theory, the degree of generality, or at least its usefulness, remains fortunately quite wide.

Indeed it will be shown in Remark 2.1 that an example of *pss* is given by the projections and normal states of a W^* algebra. From this it follows that also the case of the C^* algebras indirectly fits our axiom since every C^* algebra $\mathcal{Q}$ can be canonically embedded in its double dual $\mathcal{Q}^{**}$ [which is a W^* algebra (Sakai, 1971)] with the result that the states of $\mathcal{Q}$ become exactly the normal states of $\mathcal{Q}^{**}$.

From axiom A1 there also follows $S_1(\bigvee_\alpha a_\alpha)\supset\bigcup_\alpha S_1(a_\alpha)$, and here the inclusion is in general strict (see for instance the standard Hilbert model in Section 3). Since the proposition $\bigvee_\alpha a_\alpha$ is defined by $\bigwedge\{x\in L\colon x\geqslant a_\alpha\ \forall\alpha\}$, it can be interpreted as the least proposition whose tests give the answer yes with certainty when at least a test of an a_α gives the answer yes with certainty.

The assumption that this holds for an arbitrary family, namely, the assumption that L is complete, is clearly an idealization. If we define

$$S_0(a)=\{s\in S\colon s(a)=0\}\qquad(a\in L)$$

from the additivity of the states we get $S_1(a^\perp)=S_0(a)$ for every $a\in L$. Hence any test of $a^\perp$ is obtained by interchanging the outcomes of a test of a.

Now let a and b be two orthogonal propositions ($a\leqslant b^\perp$). We have $S_1(a)\subset S_0(b)$ so that the answer yes of a test of a implies the answer no for a test of b (and conversely). From the statistical interpretation it should then hold $s(a\vee b)=s(a)+s(b)$ $\forall s\in S$, which has indeed been assumed by A3. This enables us also to interpret the orthomodularity of L. If $s\in S$ and $a, b\in L$, $a\leqslant b$, we have $s(b)=s(a)+s(a^{\perp}\wedge b)$ by the additivity of the states and by the orthomodularity of L. Hence the physical system behaves "classically" as far as the observables a and b are considered.

The assumption of the convexity of S (A4) is motivated by the fact that the statistical mixtures of preparing procedures of Σ define new preparing procedures of Σ.

The role played by the axiom A5, which is essentially a technical condition, will be clear in the proof of some results of the paper.

For completeness we remark that the propositions ϕ and $\mathbb{1}$ do not represent rigorously classes of yes–no experiments on Σ. Indeed the corresponding "tests" have only one outcome: $S_1(\mathbb{1})=S_0(\phi)=S$ and $S_0(\mathbb{1})=S_1(\phi)=$ empty set. By forcing the given physical interpretation we assume that $\mathbb{1}$ and ϕ represent the "tests" of the existence or of the nonexistence of Σ, respectively.

We now take into consideration the fact that the physical system Σ may be a classical, a purely quantum, or a quantum system.

A mathematical classification in this sense may be obtained by the following arguments. We have seen that the orthomodular condition of L implies a classical behavior of two propositions a, b such that $a\leqslant b$. It is a result in the theory of the orthocomplemented lattices that the orthomodular condition is equivalent to the condition: if $a\leqslant b$ then the lattice generated by the family $\{a, a^{\perp}, b, b^{\perp}\}$ is distributive (Maeda and Maeda, 1970). It is possible, however, that the last family generates a distributive lattice even if the condition $a\leqslant b$ is not satisfied. In an orthomodular lattice this happens if and only if a commutes with b (written aCb), namely, if $a=(a\wedge b)\vee(a\wedge b^{\perp})$.

If now s is a state and aCb in the logic of a *pss* (L, S) of the physical system Σ we have $s(a)=s(a\wedge b)+s(a\wedge b^{\perp})$ since $a\wedge b\perp a\wedge b^{\perp}$. It follows that the physical system behaves "classically" with respect to the propositions a, b if and only if aCb.

A *classical system* can then be characterized by the condition xCy $\forall x, y\in L$, or equivalently, by the condition $C(L)=L$.

If, on the other hand, xCy is not satisfied, for every y in L, by any nontrivial x, that is if $C(L)=\{\phi, \mathbb{1}\}$ (L is irreducible), we say that Σ is a *purely quantum system*. If $C(L)$ is strictly contained in L and possesses nontrivial propositions, we say that Σ is a *quantum system endowed with superselection rules*.

By using standard results in the lattice decomposition theory we have some more information in the last case. Indeed let $\{z_\alpha\colon\ \alpha\in I\}$ be any central decomposition of $\mathbb{1}$, that is, any family of elements of $C(L)$ such that

(i) $z_\alpha \wedge z_\beta = \phi$ if $\alpha \neq \beta$ (and hence $z_\alpha \perp z_\beta$)
(ii) $\bigvee_\alpha z_\alpha = \mathbb{1}$

Then L can be decomposed into the direct sum (even continuous):

$$L = \dot{\bigvee}^{\oplus}(L[\phi, z_\alpha] : \alpha \in I)$$

of the relatively orthocomplemented logics $L[\phi, z_\alpha] = \{x \in L: \phi \leqslant x \leqslant z_\alpha\}$ (*superselection sectors*) which come out to be mutually orthogonal since $x \in L[\phi, z_\alpha]$, $y \in L[\phi, z_\beta]$ imply $x \leqslant z_\alpha \leqslant z_\beta^\perp \leqslant y^\perp$.

If a superselection sector turns out to be irreducible, then, with respect to its propositions, the physical system behaves as a purely quantum system.

It may also happen that a decomposition of L exists such that all the superselection sectors are irreducible and hence with respect to each of them Σ behaves as a purely quantum system which is different, a priori, from sector to sector. This is the case in which $C(L)$ is an atomic logic, by using the atoms $A(C(L))$ of $C(L)$ to decompose $\mathbb{1}$:

$$L = \dot{\bigvee}^{\oplus}(L[\phi, z] : z \in A(C(L)))$$

Indeed if $z \in A(C(L))$ then $L[\phi, z]$ is irreducible (Varadarajan, 1968, Theorem 6.19).

An interesting subcase is that in which the superselection sectors are all orthoisomorphic to a given logic $L^\wedge$ of a purely quantum system $\Sigma^\wedge$. If $C(L)$ is interpreted as the logic of a classical system Σ_1, then L could be interpreted as the result of a possible coupling of the logics $C(L)$ and $L^\wedge$ and correspondingly Σ as the compound system $\Sigma^\wedge + \Sigma_1$. The proposed interpretation is motivated by the fact that, as has been shown in Zecca (1978) and Aertz and Daubechies (1978), there are cases of interest where a definition of product of classical and quantum logics makes sense which gives rise to a quantum logic with continuous superselection rules. The associated superselection sectors come out to be orthoisomorphic images of the original purely quantum logic while the center of the product logic turns out to be orthoisomorphic to the original classical logic.

Remark 2.1. There are interesting example of pairs (L, S) which fit our axioms A1–A5. Let M be a W^* algebra. It is a standard result that M can be faithfully represented by a von Neumann algebra of bounded operators on some Hilbert space H (H not necessarily separable) (Sakai, 1971). If we denote by M^p the projections of M and by N the normal states of M restricted to M^p, then *the pair* (M^p, N) *is a pss*. Indeed M^p is a complete orthomodular lattice with respect to the order relation $p, q \in M^p$ $p \leqslant q \Leftrightarrow q-p \geqslant 0$ and to the orthocomplementation $p \to p' = 1-p$ (Sakai, 1971). Moreover we have the formula

$$S_1(p) = \{\varphi \in N: \operatorname{supp}\varphi \leqslant p\} \qquad (p \in M^p)$$

where $\operatorname{supp}\varphi$ is the support of the normal state φ, namely, the least projection of M^p which takes the value 1 on φ (Sakai, 1971). By taking into account the additivity of the normal states we have then $p, q \in M^p p \leqslant q \Rightarrow S_1(p) \subset S_1(q)$. To show the converse, let φ_x be the normal state defined by $\varphi_x(.) = (x, .x)$ with $x \in pH$, $\|x\| = 1$. It holds $\varphi_x(p) = 1$. If now $S_1(p) \subset S_1(q)$ we have also $\varphi_x(q) = 1$, which can be written $(x, (1-q)x) = 0$ $\forall x \in pH$. Hence $(1-q)py = 0$ $\forall y \in H$, and then $qp = p$ or $p \leqslant q$. We have thus checked the validity of axiom A1. As to A2 we have identically

$$S_1(\wedge_\alpha p_\alpha) = \{\varphi \in N: \operatorname{supp}\varphi \leqslant p_\alpha \ \forall_\alpha\} = \bigcap_\alpha S_1(p_\alpha) \qquad (\{p_\alpha\} \subset M^p)$$

The remaining axioms hold trivially.

A first limiting case of the previous situation is that in which $(L, S) = (L(H), K(H))$, $L(H)$ being the irreducible logic of the closed subspaces of a separable complex Hilbert space H and $K(H)$ the convex set of the positive trace class operators on H with trace 1.

The probabilities are given here by the formula $\rho(a) = \operatorname{Tr} P^a \rho$, $\rho \in K(H)$, $a \in L(H)$, P^a being the orthogonal projection on H with range a.

Another limiting case is the one in which M is a commutative W^* algebra. In this case M^p comes out to be a distributive logic and hence it can be associated to some classical physical system.

Remark 2.2. Axiom A2 of Definition 2.1 could seem too restrictive when applied to the case of the W^* algebras. Indeed such an axiom seems to rule out of the scheme the states that are not normal. This fact is, however, weakened not only because of the previously mentioned relation between a C^* algebra and its double dual, but also by the fact that, in many cases, the choice itself of the normal states is not too limiting on physical grounds.

In this connection let us consider directly the case of the C^* algebras which are used in physics in statistical mechanics as well as in relativistic quantum field theory (Ruelle, 1969; Haag and Kastler, 1964).

If $\mathscr{A}$ is a C^* algebra and $\{\pi_\alpha\}$ is a family of representations of $\mathscr{A}$ then every element of $\{\pi_\alpha\}$ can be considered as a subrepresentation of the same representation ρ of $\mathscr{A}$ defined by $\rho = \oplus \pi_\alpha$ (Dixmier, 1969).

If $\{\varphi_\alpha\}$ are the states of $\mathcal{A}$ which are of physical interest and π_{φ_α} are the corresponding GNS representations (Sakai, 1971)

$$\varphi_\alpha(a)=(\xi_\alpha, \pi_\alpha(a)\xi_\alpha), \qquad a\in\mathcal{A}$$

then all the states φ_α are normal when extended on

$$M=\rho(\mathcal{A})''$$

the von Neumann algebra generated by $\rho(\mathcal{A})$. By the considerations of the previous Remark we can then use as a *pss* to describe the physical situation the projections and the normal states of $\rho(\mathcal{A})''$.

As a special case the family $\{\varphi_\alpha\}$ could be the set of all the states of $\mathcal{A}$: the algebra $\rho(\mathcal{A})''$ is then isomorphic to the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ (Sakai, 1971).

Another special situation is that in which the von Neumann algebras $\pi_\varphi(\mathcal{A})''$ and $\pi_\psi(\mathcal{A})''$ generated by the GNS representations corresponding to the states φ and ψ are *factors* [these kinds of states are used in physics to represent pure phases in statistical mechanics (Hugenholtz, 1967)]. Only the following two mutually exclusive cases can then happen (Dixmier, 1969):

(a) The set of states of $\mathcal{A}$ which are normal on $\pi_\varphi(\mathcal{A})''$ coincides with the states of $\mathcal{A}$ which are normal on $\pi_\psi(\mathcal{A})''$; in this case the two representations are quasiequivalent, so that $\pi_\varphi(\mathcal{A})''$ is isomorphic to $\pi_\psi(\mathcal{A})''$. We can then choose as a *pss* the projections and the normal states of $\pi_\varphi(\mathcal{A})''$.

(b) π_φ and π_ψ are disjoint representations, so that for every representation π such that $\varphi(a)=(\Phi, \pi(a)\Phi)$ and $\psi(a)=(\Psi, \pi(a)\Psi)$ it holds (Hepp, 1972)

$$(\lambda\Phi+\mu\Psi|\pi(a)|\lambda\Phi+\mu\Psi)=|\lambda|^2\varphi(a)+|\mu|^2\psi(a), \qquad a\in\mathcal{A}.$$

We can then choose the representation $\pi=\pi_\varphi\oplus\pi_\psi$ defined by $\pi(a)=\pi_\varphi(a)+\pi_\psi(a)$ $(a\in\mathcal{A})$ in $H_\pi=H_\varphi\oplus H_\psi$. The description in terms of a *pss* can be done by using the projections and the normal states of $\pi(\mathcal{A})''$.

2.2. States and Ideals. We now discuss some concepts that will be very useful in the following. The states of a *pss* (L, S) can be used to introduce ideals in the logic L. For every nonempty subset D of S we define the subsets of L:

$$L(D)=\{\alpha\in L: D\subset S_1(a)\}$$

$$O(D)=\{a\in L: D\subset S_0(a)\}$$

The above sets have the following properties:

(i) $a \in L(D)$, $b \in L$, $a \leqslant b \Rightarrow b \in L(D)$ (by A1)
(ii) $a, b \in L(D) \Rightarrow a \wedge b \in L(D)$ (by A2)
(iii) $\wedge L(D) \neq \phi$ and $\wedge L(D) \in L(D)$ [indeed $D \subset S_1(\wedge L(D)) = \bigcap_{x \in L(D)} S_1(x)$]

There follows that $L(D)$ is a dual principle ideal of the logic L (Birkhoff, 1967).

This means that there exists an $a \in L$ such that

$$L(D) = \{x \in L: x \geqslant a\}$$

[such a turns out to be exactly $a = \wedge L(D)$].

The set $O(D)$ has the dual properties (in the sense of the lattice theory) of those of $L(D)$, and hence it is a principal ideal of L which can be represented as

$$O(D) = \{x \in L: x \leqslant \vee O(D)\}$$

We state now some results for the proof of which we refer to Berzi and Zecca (1974).

Lemma 2.1. Let (L, S) be a *pss*. Then

(i) $\wedge L(D) = \bigvee_{s \in D} (\wedge L(s)) \qquad \forall D \subset S$

(ii) $s = \sum_1^N \alpha_i s_i$, $\{\alpha_i\} \subset [0, 1]$, $\sum_1^N \alpha_i = 1$, $\{s_i\} \subset S \Rightarrow L(s) = \bigcap_{i=1}^N L(s_i)$

where $L(s)$ stands for $L(\{s\})$.

To complete the section we recall that a (nontrivial) dual principle ideal I is said to be *maximal* if $I' \supset I$, I' a (nontrivial) dual principle ideal, implies $I' = I$. A maximal dual principle ideal has the form

$$I = \{x \in L: x \geqslant e\}$$

where e is an atom of L. There is then a one-to-one correspondence between the atoms of L and the maximal dual principle ideals of L.

Remark 2.3. In the case of the projections and normal states (M^p, N) of a W^* algebra M we get the formulas

$$L(\varphi)=\{p\in M^p; \operatorname{supp}\varphi\leqslant p\}, \qquad \varphi\in N$$

$$\wedge L(\varphi)=\operatorname{supp}\varphi$$

$$\wedge L(D)=\bigvee_{\varphi\in D}\operatorname{supp}\varphi, \qquad D\subset N.$$

2.3. Characteristic States and Atomicity Condition. The request of the atomicity of the logic associated to the physical system is generally assumed when one has in mind to specialize the axiomatic scheme to get the standard Hilbert model. On the basis of the physical interpretation of the *pss* (L, S) which has been associated in Section 2.1 with the physical system, such a condition cannot be required in general. There are indeed examples of physical interest where the logic L not only is not atomic, but it does not even contain any atomic proposition.

Remark 2.4. Let us consider the *pss* (M^p, N) of Remark 2.1. We show that *if M is a type II or type III W^* algebra, then M^p has no atoms*. Indeed suppose a to be an atom of M^p and consider the linear positive map $x\to axa$ from M to the W^* algebra aMa. If x is a positive element of M, $x\leqslant\|x\|\cdot 1$ and hence $axa\leqslant\|x\|a$. In particular if $x=aya$ is a positive element of M we have $x=axa\leqslant\|x\|a$ so that if x is a projection of aMa we have $x=0$ or $x=a$. It follows that aMa contains no projections except a so that aMa is an Abelian algebra (a is an Abelian projection) and hence a is a finite projection (Sakai, 1971, Proposition 2.2.8). This is a contradiction since a type-II or -III W algebra has no finite projections.

The situation of the previous Remark is not a purely mathematical example. It refers to a precise physical situation. Indeed, in quantum statistical mechanics, a way to study a thermal equilibrium state φ for a physical system Σ is that of considering the von Neumann algebra M generated by the C^* algebra of the quasilocal observables of Σ on the Hilbert space of the GNS representation defined by φ and to consider the normal extension of φ to M. If φ is the thermal equilibrium state corresponding to one phase of a free Bose gas, then M is a type-III von Neumann algebra (Hugenholtz, 1967).

Even if the logic L is not in general atomic, it is still possible that L contains some atomic propositions. Such atoms of L can be characterized in alternative ways by using the concept of lattice ideal introduced in the previous section.

Definition 2.3. A maximal state of a *pss* (L, S) is an $s \in S$ such that $L(s)$ is a maximal (dual principle) ideal. The set of the maximal states will be denoted by S_m.

If $s \in S_m$ we have that $\wedge L(s) \in A(L)$ (the atoms of L). Conversely, if $a \in A(L)$ and $s \in S_1(a)$ then $\wedge L(s) = a$. It follows that the correspondence $s \to \wedge L(s)$ between maximal states and atoms of L is onto (even if, a priori, not one to one). A class of states that are both pure and maximal (as will be presently seen) can be characterized in the following way.

Definition 2.4. A *characteristic state* of a *pss* (L, S) is an $s \in S$ such that $s' \in S$, $L(s') = L(s) \Rightarrow s = s'$. The set of the characteristic states will be denoted by S_c.

Proposition 2.1. Let (L, S) be a *pss*. Then

$$S_c \subset S_m \cap S_p$$

Proof. (i) $S_c \subset S_m$: let $s \in S_c$, then $s \in S_1(\wedge L(s))$. If $s' \in S_1(\wedge L(s))$, $s \neq s'$, then $L(s') \supset L(s)$. Consider now $\bar{s} = \alpha s' + (1-\alpha)s$, $\alpha \in [0,1]$. By Lemma 2.1 we have $L(\bar{s}) = L(s)$ and hence $\bar{s} = s$, s being characteristic. (ii) $S_c \subset S_p$: suppose $s \in S_c$ and $s = \alpha s_1 + (1-\alpha)s_2$. We have $L(s) = L(s_1) \cap L(s_2)$. By Lemma 2.1, $\wedge L(s) = (\wedge L(s_1)) \vee (\wedge L(s_2))$. Since, by (i), $\wedge L(s)$ is an atom, it must be $\wedge L(s) = \wedge L(s_1) = \wedge L(s_2)$ or $L(s) = L(s_1) = L(s_2)$ and hence $s = s_1 = s_2$, s being characteristic.

A characteristic state is then determined by the set of propositions that are true with certainty on it, or, equivalently, by the property $S_1(\wedge L(s)) = \{s\}$ (which implies, by A1, that $\wedge L(s)$ is an atomic proposition).

The results just obtained suggest a sufficient condition for the atomicity of the logic of the physical system.

Proposition 2.2. Let (L, S) be a *pss* such that $\forall a \in L$, $a \neq \phi$; then at least one of the following conditions holds: (i) a is an atom; (ii) $S_1(a) \cap S_c$ is different from the empty set.

Then L is an atomic logic. Indeed if a is not an atom and $a \neq \phi$, from $s \in S_1(a) \cap S_c$ we have $a \in L(s)$ and hence $\wedge L(s)$ (which is an atom by Proposition 2.1) is such that $\wedge L(s) \leqslant a$.

Roughly speaking, the atomicity of L is ensured by the requirement of the existence of sufficiently many characteristic states.

We remark that the atomicity of L could be obtained, instead of the assumption of Proposition 2.2, with the condition

$$a \in L \Rightarrow S_1(a) \cap S_m \neq \text{empty set}$$

since also in this case we have that $\bigwedge L(s)$ is an atom such that $\bigwedge L(s) \leqslant a$ for every $s \in S_1(a) \cap S_m$. However, the condition given in terms of characteristic states should be preferred since the maximal states are not, a priori, completely determined by the propositions certainly true on them.

2.4. Characteristic, Pure, and Maximal States. A problem to which Proposition 2.1 gives rise is that of establishing whether there are pure or maximal states which are not characteristic; whether the maximal states are pure or, conversely, if the pure states are maximal states.

We will not be able to give a complete answer to that question at the level of a general proposition-state structure (L, S). We are able to solve the problem in three interesting cases: when the logic L has a continuous center; when L is the logic of a classical physical system, and in the example of the W^* algebras.

Proposition 2.3. Let (L, S) be a *pss* such that $C(L)$ is continuous, namely, $\forall c \in C(L)$, $\exists c_1 \neq \phi, c\ c_1 \in C(L)$ such that $c_1 \leqslant c$. Then

$$S_c = S_m = S_p = \text{empty set of states}$$

Proof. We first show that S_p is empty. By mimicking the proof of Theorem 6.19 of Varadarajan (1968), it is possible to show that, also with our assumptions, for every $s \in S$ there exists a $d \in C(L)$ such that $0 < s(d) < 1$. By Axiom A5 we have $s_d, s_{d^\perp} \in S$ with $s_d \neq s_{d^\perp}$. Then s can be decomposed as

$$s = s(d)s_d + (1 - s(d))s_{d^\perp}, \qquad d \text{ being in } C(L)$$

so that s is not a pure state. We now show that S_m is empty by showing that $A(L)$ is empty. Suppose $a \in A(L)$ and let $e(a)$ be the central cover of a.

If $x \leqslant e(a)$, $x \neq e(a)$, $x \in C(L)$ then $x \wedge a = \phi$. It follows $\phi = e(x \wedge a) = x \wedge e(a) = x$ so that $e(a)$ is an atom of $C(L)$. But this is not possible since L has a continuous center by assumption. The proof is completed since $S_c \subset S_p \cap S_m$ by Proposition 2.1.

Proposition 2.4. Let (L, S) be a *pss* such that $C(L) = L$. Then

(i) $S_c = S_p = S_m$
(ii) $s \in S_p \Rightarrow s(x) = 1, 0\ \forall x \in L$

Proof. If $s \in S_m$, denote $\bigwedge L(s) = e\ (\in A(L))$. If $x \in L$ we have $x = (x \wedge e) \vee (x \wedge e^\perp)$, L being distributive. Hence $s(x) = 0, 1$ since either $x \geqslant e$ or $x \leqslant e^\perp$. This implies also $s \in S_c$. By Proposition 2.1 we have then shown

$S_c = S_m \subset S_p$. Suppose now $s \in S_p$. If there is an $a \in L$ such that $0 < s(a) < 1$ we have $s_a, s_{a^\perp} \in S$ (by A5) and s can be decomposed as $s = s(a)s_a + (1 - s(a))s_{a^\perp}$, which is absurd, s being a pure state. Hence $s(x) = 0, 1$ $\forall x \in L$.

This implies that $\bigwedge L(s)$ is an atom. Indeed if $e \leqslant \bigwedge L(s)$, $e \neq \bigwedge L(s)$ then $s(e) = 0$ so that $e = \phi$ or $e \leqslant (\bigwedge L(s))^\perp$ and hence $e \leqslant (\bigwedge L(s)) \wedge (\bigwedge L(s))^\perp = \phi$. It follows that $s \in S_m$ and the proof is completed.

Remark 2.5. Also in the example of the projections and normal states (M^p, N) of a W^* algebra we have the result

$$S_c = S_p = S_m$$

so that there is a bijection between the atoms and the pure states of N. To show the result we first show that $S_p \subset S_m$. Indeed let $\{\pi_\varphi, H_\varphi, x_\varphi\}$ be the GNS representation defined by the state φ, and denote by $\tilde{\varphi}$ the action of φ in $\pi_\varphi(M)$:

$$\tilde{\varphi}(\pi_\varphi(a)) = (x_\varphi, \pi_\varphi(a)x_\varphi) = \varphi(a), \qquad a \in M$$

If φ is a pure and normal state we have (Sakai, 1971, Propositions 1.21.9 and 1.21.10)

$$\pi_\varphi(M)'' = \pi_\varphi(M) = B(H_\varphi) \equiv (\textit{all} \text{ the bounded operators in } H_\varphi)$$

and hence $\operatorname{supp} \tilde{\varphi} = |x_\varphi\rangle\langle x_\varphi| \in \pi_\varphi(M)$. We distinguish two cases:

(i) π_φ *is a faithful representation*: then $\pi_\varphi^{-1}(|x_\varphi\rangle\langle x_\varphi|) = \operatorname{supp} \varphi$ is an atom in M^p.

(ii) π_φ *is not a faithful representation*. Suppose then that $\operatorname{supp} \varphi$ is not an atom in M^p, namely, that there is a $q \in M^p$, $q \leqslant \operatorname{supp} \varphi$, $q \neq \operatorname{supp} \varphi$. Then it must be the case that either

(a) $$\pi_\varphi(q) = \pi_\varphi(\operatorname{supp} \varphi) = |x_\varphi\rangle\langle x_\varphi|$$

or

(b) $$\pi_\varphi(q) = 0 \qquad \text{since } |x_\varphi\rangle\langle x_\varphi| \text{ is an atom in } \pi_\varphi(M)$$

If (a) holds we have that $\varphi(q) = 1$, which implies $q \geqslant \operatorname{supp} \varphi$ and hence $\operatorname{supp} \varphi$ is an atom of M^p. If (b) holds we have $\varphi(\operatorname{supp} \varphi - q) = \varphi(\operatorname{supp} \varphi) = 1$, which implies $\operatorname{supp} \varphi - q \geqslant \operatorname{supp} \varphi$, and this is impossible unless $q = 0$. Both in (i) and (ii) we have thus shown that $\operatorname{supp} \varphi$ is an atom when φ is a normal pure state and hence that φ is a maximal state.

By taking into account Proposition 2.1 we have thus shown

$$S_c \subset S_p \subset S_m$$

To complete the proof we show that a maximal state is a characteristic state. Let indeed $\varphi \in N$ with $\operatorname{supp}\varphi = \bigwedge L(\varphi)$ an atom in M^p. If $\psi \in N$ is such that $L(\varphi) = L(\psi)$ and hence $\operatorname{supp}\psi = \operatorname{supp}\varphi$, by the very definition of support of a normal state we have

$$\varphi(a) = \varphi(\operatorname{supp}\varphi \cdot a \cdot \operatorname{supp}\varphi)$$

$$\psi(a) = \psi(\operatorname{supp}\varphi \cdot a \cdot \operatorname{supp}\varphi) \qquad \forall a \in M$$

Since $\operatorname{supp}\varphi$ is an atom, $\operatorname{supp}\varphi \cdot M \cdot \operatorname{supp}\varphi$ is an Abelian algebra having no projections except $\operatorname{supp}\varphi$ (see Remark 2.4) and hence $\operatorname{supp}\varphi \cdot a \cdot \operatorname{supp}\varphi = \alpha(a) \cdot \operatorname{supp}\varphi$ $\forall a \in M$, $\alpha(a)$ being a complex number.

By using this result in the last two equations we get

$$\psi(a) = \varphi(a) = \alpha(a) \qquad \forall a \in M, \text{ that is } \psi = \varphi$$

We want now to study the effect of the possible superselection rules of the physical system Σ on the structure of the pure states.

If (L, S) is a *pss* and $z \in L$, we denote by $\tilde{S}_1(z)$ the restriction of $S_1(z)$ to the relatively orthocomplemented logic $L[\phi, z]$ (with the relative orthocomplementation $x \to x' = x^\perp \wedge z$). We also denote by $\tilde{s}$ the restriction to $L[\phi, z]$ of the element $s \in S_1(z)$.

Theorem 2.1. Let (L, S) be a *pss* and let $z \in C(L)$. Then (i) $(L[\phi, z], \tilde{S}_1(z))$ is a *pss*. If $\tilde{S}_p(z)$ denotes the pure states of $\tilde{S}_1(z)$, then (ii) $\tilde{s} \in \tilde{S}_p(z)$ implies $\tilde{s}^* \in S_p$, where $\tilde{s}^*$ is defined by $\tilde{s}^*(x) = \tilde{s}(x \wedge z)$, $x \in L$. (iii) For *every* central decomposition $\{z_\alpha\}$ of $\mathbb{1}$ (see Section 2.1) by setting

$$S_\alpha = \left\{ \tilde{s}^* : \tilde{s} \in \tilde{S}_p(z_\alpha) \right\}$$

we have

(a) $$\alpha \neq \beta \Rightarrow S_\alpha \cap S_\beta = \text{empty set}$$

(b) $$S_p = \bigcup_\alpha S_\alpha$$

Proof. (i) We have to check that the convex set $\tilde{S}_1(z)$ of additive measures on $L[\phi, z]$ satisfies axioms A1–A5. This poses no problem except for axiom A5. To show the validity of that axiom we preliminarily show that if d is an element of the center of $L[\phi, z]$, then $d \in C(L)$. Indeed we first have that yCd in $L \forall y \leqslant z$ since, by the assumptions on d, $y = (y \wedge d) \vee (y \wedge d') = (y \wedge d) \vee (y \wedge (d^{\perp} \wedge z)) = (y \wedge d) \vee (y \wedge d^{\perp})$.

This implies that $(x \wedge z)Cd \ \forall x \in L$. Moreover $(x \wedge z^{\perp})Cd$ since $d \perp x \wedge z^{\perp}$ (Maeda and Maeda, 1970, Lemma 36.3). It follows that d commutes with $(x \wedge z) \vee (x \wedge z^{\perp})$ (Maeda and Maeda, 1970, Lemma 36.4) and hence $xCd \ \forall x \in L$ since $x = (x \wedge z) \vee (x \wedge z^{\perp})$, z being in $C(L)$. If now $\tilde{s} \in \tilde{S}_1(z)$ and d in the center of $L[\phi, z]$ is such that $\tilde{s}(d) \neq 0$, by setting $\tilde{s}_d(x) = \tilde{s}(x \wedge d)/\tilde{s}(d)$, $x \leqslant z$, we have that $\tilde{s}_d \in \tilde{S}_1(z)$ since $\tilde{s}_d = s_d|_{L[\phi, z]}$ and $s_d \in S$ by A5, d being in $C(L)$.

(ii) By taking into account A5 it is easy to show that $\tilde{s}^* \in S$. Suppose now

$$\tilde{s}^* = \alpha s_1 + (1-\alpha) s_2 \qquad \alpha \in [0,1], \quad s_1, s_2 \in S$$

We have $s_1(z) = s_2(z) = 1$, hence $\tilde{s}_1, \tilde{s}_2 \in \tilde{S}_1(z)$. Since $\tilde{s} \in \tilde{S}_p(z)$ we have $\tilde{s} = \tilde{s}_1 = \tilde{s}_2$. Then $s = s_1 = s_2$ since $\forall x \in L$ we have $x = (x \wedge z) \vee (x \wedge z^{\perp})(z \in C(L))$ and $s_1(x \wedge z^{\perp}) = s_2(x \wedge z^{\perp}) = 0$.

(iiia) Let $\tilde{s}^* \in S_\alpha$. Then $\tilde{s}^*(z_\alpha) = 1$ while $\tilde{s}^*(z_\beta) = \tilde{s}^*(\phi) = 0$ and hence $\tilde{s}^* \notin S_\beta$.

(iiib) We have to prove that if $s \in S_p$ then there is an α such that $s \in S_\alpha$. We first show that $s(z_\alpha) \neq 0$ for some α. Indeed if $s(z_\alpha) = 0 \ \forall \alpha$, we have the contradiction $s \in \cap_\alpha S_1(z_\alpha^{\perp}) = S_1(\wedge_\alpha z_\alpha^{\perp}) = S_1((\vee_\alpha z_\alpha)^{\perp}) = \text{empty set}$. Secondly if $0 < s(z_\alpha) < 1$ we could write, by taking into account also A5,

$$s = s(z_\alpha) s_{z_\alpha} + (1 - s(z_\alpha)) s_{z_\alpha^{\perp}}$$

so that s would be not pure. Hence $s(z_\alpha) = 1$ for some α so that $\tilde{s} \in \tilde{S}_1(z_\alpha)$. We now show that $\tilde{s}$ is a pure state in $\tilde{S}_1(z_\alpha)$. If $\tilde{s} = \gamma \tilde{s}_1 + (1-\gamma)\tilde{s}_2$, $\alpha \in [0,1]$, then, by setting $\tilde{s}^*(x) = \tilde{s}(x \wedge z_\alpha)$, $\tilde{s}_1^*(x) = \tilde{s}_1(x \wedge z_\alpha)$, $\tilde{s}_2^*(x) = \tilde{s}_2(x \wedge z_\alpha)$ we have $\tilde{s}^* = \gamma \tilde{s}_1^* + (1-\gamma)\tilde{s}_2^*$. But $\tilde{s}^*(x) = s(x)$ since from $x = (x \wedge z_\alpha) \vee (x \wedge z_\alpha^{\perp})$ and $s(x \wedge z_\alpha^{\perp}) = 0$ we have $s(x) = s(x \wedge z_\alpha) = \tilde{s}(x \wedge z_\alpha)$. It follows that $s = \gamma s_1 + (1-\gamma) s_2$, which implies $s = s_1 = s_2$, since s is a pure state, and hence $\tilde{s} = \tilde{s}_1 = \tilde{s}_2$.

We remark that the results (ii) and (iii) of Theorem 2.1 are generalizations to our scheme of results obtained by Varadarajan (1968), where, however, the further assumption of atomicity of the center of the logic has been made. That situation corresponds to the special case of Theorem 2.1 in which $C(L)$ is atomic and the atoms of $C(L)$ have been chosen to have the central decomposition of $\mathbb{1}$.

2.5. Superposition and Closure Under Superposition. We now discuss the superposition relation for the states of the physical system. The language of the lattice theory will be very useful to that end.

The definition of the superposition relation we use concerns all the states and, in the Hilbert model, when it is restricted to the pure states it is equivalent to Dirac's superposition of pure states (see Section 3.1).

Moreover it furnishes a unified formulation of the superposition in the sense that it contains the concept of quantum superposition as well as the concept of statistical mixture. The definition we use is equivalent, in our scheme, to the one originally introduced by Varadarajan (1968) and studied by other authors (Gudder, 1970; Pulmannovà, 1976; Berzi and Zecca, 1974; Gorini and Zecca, 1975).

Definition 2.5. Let (L, S) be a *pss* associated to the physical system. We say that a state s is a *superposition* of the states in $D \subset S$ if $L(s) \supset L(D)$.

The physical interpretation is the following. Suppose that a yes–no experiment on the system Σ gives the answer yes with certainty when Σ has been prepared with any one of the preparing procedures corresponding to the states of D. Then to say that s is a superposition of the states of D means that the probability of the outcome yes for the same yes–no experiment performed on Σ is 1 once Σ has been prepared according to s. Equivalently, the states being additive measures on the logic, s is a superposition of the states of D if $s'(a)=0$ $\forall s' \in D$ $(a \in L)$ implies $s(a)=0$, which is just the definition introduced by Varadarajan.

Since $L(s)$ and $L(D)$ are dual principle ideals (Section 2.2) we have that $L(s) \supset L(D)$ if and only if $\wedge L(s) \leqslant \wedge L(D)$. By taking into account Lemma 2.1, it follows that s is a superposition of the states of D if and only if

$$\wedge L(s) \leqslant \bigvee_{s' \in D} (\wedge L(s')$$

If as in Lemma 2.1, $s = \Sigma_1^N \alpha_i s_i$ we have $L(s) = L(\{s_i\})$ so that the statistical mixtures are a limiting case of the superposition relation.

We now introduce a closure operation in the subsets of S. For every $D \subset S$ define

$$\bar{D} = \bigcap_{S_1(x) \supset D} S_1(x) \qquad (x \in L)$$

In Gorini and Zecca (1975) it has been shown that the map $D \to \bar{D}$ is a closure operation on the subsets of S and that

$$\bar{D} = \{s \in S \colon L(s) \supset L(D)\} = S_1(\wedge L(D))$$

The last result motivates the name of *closure under superposition* for the map $D \to \bar{D}$ and has the direct consequence that the subsets of S which are closed under superposition are exactly the $S_1(a)$'s, $a \in L$.

Remark 2.6. In the case of the *pss* (M^P, N) of the W^* algebra model we have that the normal state φ is a superposition of the normal states in $D \subset N$ if and only if

$$\operatorname{supp}\varphi \leqslant \bigvee_{\psi \in D} \operatorname{supp}\psi$$

and that the closure under superposition is given by (see Remark 2.3)

$$\bar{D} = \left\{ \varphi \in N: \operatorname{supp}\varphi \leqslant \bigvee_{\psi \in D} \operatorname{supp}\psi \right\}$$

2.6. The Superposition Principle for the Pure States. In Section 2.4 it has been shown that in many interesting situations the concepts of maximal, pure, and characteristic state coincide even if we have not been able to solve the problem in general.

Hereafter we restrict our considerations to the class of *pss* (L, S) for which the relation $S_c = S_p = S_m$ holds. There will be then in the *pss* we will consider a bijection between the atoms of L and the pure states of S.

It is well known that a point where the difference between classical and quantum physical theories is more evident is the one concerning the superposition of pure states. Roughly speaking, while for a quantum physical system it is possible to construct new pure states by using pure states, for a classical system the only pure superpositions of pure states are the trivial ones (Gudder, 1970; Varadarajan, 1968).

Definition 2.6. Let (L, S) be a *pss* of the physical system Σ. We say that a *classical superposition principle* holds for (L, S) if $s \in S_p$, $D \subset S_p$ and $L(s) \supset L(D) \Rightarrow s \in D$.

The following results show that the classical physical systems are the natural context for the validity of a classical superposition principle.

Proposition 2.5. Let (L, S) be a *pss* such that $C(L) = L$. Then a classical superposition principle holds for (L, S).

Proof. If $L(s) \supset L(D)$, $s \in S_p$, $D \subset S_p$, we have $\wedge L(s) \leqslant \vee_{s' \in D}(\wedge L(s'))$ with $\wedge L(s), \wedge L(s')$ $\forall s' \in D$ atoms of L. Since $C(L)$ is a complete lattice it is a completely distributive lattice (Crawley and Dilworth, 1973). Hence

$$\wedge L(s) = \wedge L(s) \wedge \Big(\bigvee_{s' \in D} \wedge L(s') \Big) = \bigvee_{s' \in D} \left[(\wedge L(s)) \wedge (\wedge L(s')) \right]$$

so that $\bigwedge L(s) = \bigwedge L(s')$ for some $s' \in D$ and hence $s = s'$ since we are considering *pss* such that $S_c = S_p = S_m$.

To show the converse result of that of Proposition 2.5 we need a preliminary Lemma which characterizes the distributivity of the logic and whose proof can be found in Berzi and Zecca (1974).

Lemma 2.2. Let L be a complete, atomic, orthocomplemented lattice. Then L is distributive if and only if

$$e \in A(L), \qquad B \subset A(L), \qquad e \leqslant \bigvee B \Rightarrow e \in B$$

Proposition 2.6. Let (L, S) be a *pss* such that L is an atomic logic. Then L is a distributive logic if a classical superposition principle holds for (L, S).

Proof. It suffices to prove that the condition of Lemma 2.2 is satisfied. Let indeed $e \in A(L)$, $\{e_\alpha\} \subset A(L)$ and consider $\{s\} = S_1(e)$, $\{s_\alpha\} = S_1(e_\alpha)$. If $e \leqslant \bigvee_\alpha e_\alpha$ it follows that $L(s) \supset \bigcap_\alpha L(s_\alpha) = L(\{s_\alpha\})$. If a classical superposition principle holds we have $s = s_\alpha$ for some α so that $e = e_\alpha$ for some α, the maximal states being assumed to be characteristic.

We now consider the quantum superposition principle of physical states. As has been done by Dirac, such a principle could even be assumed as the starting point of the quantum theory.

In the standard formulation of quantum mechanics the pure states are represented by rays of the Hilbert space H of the physical system. In this context the quantum superposition principle is expressed by the fact that new pure states can be obtained by linearly combining representative vectors of given rays and by normalizing to 1 the resulting vectors. Since there is a bijection between rays of H and atoms (one-dimensional subspaces) of the logic $L(H)$ of the closed subspaces of H, it is obvious that for every pair e_1, e_2 of atoms of $L(H)$ there is a third atom of $L(H)$ which is different from e_1, e_2 and which is such that

$$e_1 \vee e_2 = e_1 \vee e_3 = e_2 \vee e_3 \tag{2.1}$$

(by $e_1 \vee e_2$ we mean the linear span in H of the vectors of e_1 and e_2).

In the logic approach to quantum mechanics where, a priori, no linear spaces are present, Jauch (1968) formulated the validity of a quantum superposition principle by just assuming the previous condition (2.1), namely, by assuming that for every pair e_1, e_2 of atoms of the logic of the physical system there is a third atom e_3 of the logic such that the relation (2.1) is satisfied. The atomic proposition e_3 which satisfies $e_3 \leqslant e_1 \vee e_2$ is said to be one of the superpositions of e_1 and e_2.

As far as the author knows, the other formulations of the quantum superposition principle are essentially variants or weakened forms of Jauch's one, both when given at the level of the logic and the level of the states (Gudder, 1970; Pulmannovà, 1976; Berzi and Zecca, 1974). This is true also for the formulation in Chen (1973), which, however, is done in terms of propositions not necessarily atomic.

Definition 2.7. Let (L, S) be a *pss* of the physical system Σ. We say that a quantum superposition principle holds for (L, S) if for every $s_1, s_2 \in S_p$ there is an $s \in S_p$, $s \neq s_1, s_2$ such that $L(s) \supset L(s_1) \cap L(s_2)$.

We now show that the physical systems for which a quantum superposition principle holds are the purely quantum physical systems, if a sufficient condition is assumed.

Proposition 2.7. If a quantum superposition principle holds for the *pss* (L, S) with L an atomic logic, then L is an irreducible logic.

Proof. By proceeding as in the first part of the proof of Proposition 2.4 one can show that a pure state takes on the elements of the center of L only the values 0, 1 even if L is not distributive.

Let now $c \in C(L)$, $c \neq \phi, \mathbb{1}$, $s_1 \in S_p$ such that $s_1(c) = 1$ and $s_2 \in S_p$ such that $s_2(c) = 0$. (This is possible since L has been assumed to be atomic.) Hence the atoms $e_1 = \bigwedge L(s_1)$, $e_2 = \bigwedge L(s_2)$ are such that $e_1 \leqslant c$, $e_2 \leqslant c^{\perp}$. If now $s \in S_p$, $s \neq s_1, s_2$ and $L(s) \supset L(s_1) \cap L(s_2)$ we have $e = \bigwedge L(s) \leqslant e_1 \vee e_2$ and $e \neq e_1, e_2$. Two cases are then possible: (i) $s(c) = 1$, that is $e \leqslant c$. Hence $e \leqslant c \wedge (e_1 \vee e_2) \leqslant c \wedge (e_1 \vee c^{\perp}) = e_1$, which is a contradiction. (ii) $s(c) = 0$, that is $e \leqslant c^{\perp}$. Hence $e \leqslant c^{\perp} \wedge (e_1 \vee e_2) \leqslant c^{\perp} \wedge (e_2 \vee c) = e_2$, a contradiction.

Quantum superpositions of pure states exist even when the physical system is not a purely quantum physical system.

Proposition 2.8. Let (L, S) be a *pss* and L a non-Boolean atomic logic. Then there is an $s \in S_p$, a $D \subset S_p$ such that $s \notin D$, and $L(s) \supset L(D)$.

Proof. By Lemma 2.2 there is an atom e of L and a family $\{e_\alpha\}$ of atoms of L such that $e \notin \{e_\alpha\}$ and $e \leqslant \bigvee_\alpha e_\alpha$. If $\{s\} = S_1(e)$, $\{s_\alpha\} = S_1(e_\alpha)$ we have that $s \notin \{s_\alpha\}$ and $L(s) \supset L(\{s_\alpha\})$.

To have further information of the effect of the superselection rules at the level of the pure states we establish some further properties relative to the atoms of the logic.

Proposition 2.9. Let L be a logic and $\{z_\alpha\}$ be a central decomposition of $\mathbb{1}$ (see Section 2.1). If we define $A(\alpha) = \{e \in A(L)\colon e \leqslant z_\alpha\}$ then we have

(i) $e \in A(L), \qquad a, b \in A(\alpha), \qquad e \leqslant a \vee b \Rightarrow e \in A(\alpha)$

(ii) $\cup_\alpha A(\alpha)=A(L)$
(iii) $A(\alpha)\cap A(\beta)=$ empty set if $\alpha\neq\beta$
(iv) $a\in A(\alpha)$, $b\in A(\beta)$ $(\alpha\neq\beta)$, $e\in A(L)$,
$e\leqslant a\vee b\Rightarrow e=a$ or $e=b$

Proof. (i) Obviously $e\leqslant a\vee b\leqslant z_\alpha$. (ii) If $e\in A(L)$ then $e=e\wedge\mathbb{1}=e\wedge(\vee_\alpha z_\alpha)=\vee_\alpha(e\wedge z_\alpha)$ (Maeda and Maeda, 1970, Lemma 29.16) and hence an α exists such that $e\leqslant z_\alpha$. (iii) If $e\in A(\alpha)\cap A(\beta)$ we have $e\leqslant z_\alpha\wedge z_\beta\leqslant z_\alpha\wedge z_\alpha^\perp=\phi$. (iv) We have first $e\leqslant z_\alpha$ or $e\leqslant z_\beta$. Indeed $e\leqslant a\vee b\leqslant z_\alpha\vee z_\beta$ implies $e=e\wedge(z_\alpha\vee z_\beta)=(e\wedge z_\alpha)\vee(e\wedge z_\beta)$ so that either $e\leqslant z_\alpha$ or $e\leqslant z_\beta$. If $e\leqslant z_\alpha$ then $e\leqslant z_\beta^\perp\leqslant b^\perp$ since $z_\alpha\leqslant z_\beta^\perp$ and $b\leqslant z_\beta$. It follows, L being orthomodular, that

$$e\leqslant b^\perp\wedge(a\vee b)=a\vee(b\wedge b^\perp)=a$$

and hence $e=a$. Analogously, $e\leqslant z_\beta$ implies $e=b$.

Definition 2.8 (Pulmannovà). We call a *sector of pure states* of a *pss* (L, S) a set $A\subset S_p$ such that

(1) $r,t\in A$, $s\in S_p$, $L(s)\supset L(r)\cap L(t)\Rightarrow s\in A$ (A is closed under pure superpositions)
(2) if $r,t\in A$, $\exists s\in A$: $L(s)\supset L(r)\cap L(t)$ (a quantum superposition principle holds in A)
(3) $t\in A$, $r\in S_p$, $r\notin A$, $s\in S_p$, $L(s)\supset L(r)\cap L(t)\Rightarrow s=r$ or $s=t$.

If a central decomposition $\{z_\alpha\}$ of $\mathbb{1}$ is given for a *pss* (L, S), by setting

$$S(\alpha)=\{s\in S_p:\ \wedge L(s)\leqslant z_\alpha\}$$

from Proposition 2.9 and the existing bijection between S_p and $A(L)$ we have immediately that $S(\alpha)$ satisfies conditions (1) and (3) of Definition 2.8. Moreover $S(\alpha)=S_\alpha$ (see Theorem 2.1) If now L is an atomic logic (so that also $L[\phi, z_\alpha]$ comes out to be atomic for every α) and a quantum superposition principle holds in $(L[\phi, z_\alpha], \tilde{S}_1(z_\alpha))$, then $L[\phi, z_\alpha]$ is irreducible (Proposition 2.7), z_α is an atom of $C(L)$, and the sectors of pure states of (L, S) are exactly the S_α's of Theorem 2.1.

2.7. Reversible Dynamics. Dealing with a reversible time evolution for the physical system Σ one can assume the Heisenberg or the Schrödinger point of view. The important fact is that the two dynamical pictures give rise to the same physical previsions.

The description of the reversible dynamical evolution of Σ can be made also in the language used in the logic approach to quantum mechanics. Indeed the Schrödinger dynamical picture is possible in terms of a one-parameter group $t\rightarrow\alpha_t$ of convex permutations of the states of Mackey (1963) and Varadarajan (1968), while the Heisenberg picture is possible by means of a one-parameter group $t\rightarrow\mu_t$ of orthoautomorphisms of the logic of the physical system (Jauch and Piron, 1969; Piron, 1976).

While, given the Heisenberg picture $t\rightarrow\mu_t$, it is possible to define directly a Schrödinger picture $t\rightarrow\alpha_t$ by setting

$$(\alpha_t s)(a)=s(\mu_t(a)), \qquad a\in L, \qquad s\in S$$

the converse problem does not seem to be obvious nor very much studied.

In connection with the mentioned problem we will assume the Schrödinger point of view. It will be seen that, also in the dynamical evolution, the superposition relation of the states has its role. The time translation invariance of the superposition relation will be one of the main elements to have physically equivalent Schrödinger and Heisenberg pictures.

Definition 2.9. A *dynamical group* of a *pss* (L,S) associated to the physical system Σ is a one-parameter group $t\rightarrow\alpha_t$, $t\in\mathbb{R}$, of convex automorphisms of S, namely, a family $\{\alpha_t\colon t\in\mathbb{R}\}$ of bijections from S onto S such that

(i) $\alpha_0=\mathbb{1}$ (the identity map in S)

(ii) $\alpha_{-t}=\alpha_t^{-1}$

(iii) $\alpha_{t+\tau}=\alpha_t\alpha_\tau$

(iv) $\alpha_t\left(\sum_1^N\gamma_i s_i\right)=\sum_1^N\gamma_i\alpha_t s_i;\qquad \gamma_i\in[0,1],\qquad \sum_1^N\gamma_i=1,\ s_i\in S$

which is *implemented*, that is such that there exists a one-parameter group $t\rightarrow\mu_t^\alpha$ of orthoautomorphisms of L such that

(v) $(\alpha_t s)(a)=s(\mu_t^\alpha(a))\qquad \forall t\in\mathbb{R},\qquad a\in L,\qquad s\in S$

and which is moreover *weakly continuous*, namely, such that the map

(vi) $$t\rightarrow(\alpha_t s)(a)$$

is continuous $\forall a\in L$, $s\in S$.

As to the physical interpretation we simply notice that condition (iv) of Definition 2.9 has been required as a consequence of the statistical interpretation of the mixtures of states.

Given a one-parameter group of permutations of the states (not necessarily convex) one could ask what condition must be added to have a dynamical group. Even if the group consists of convex permutations of the states one could ask when it happens that it is also a dynamical group. To have some answers in that direction we collect in the form of a theorem the results obtained through many steps in Gorini and Zecca (1975) and then make some applications.

Theorem 2.2. Let (L, S) be a *pss* of Σ and suppose that S is stable under orthoautomorphisms of L, namely, that if μ is an orthoautomorphism of L, by setting $\tilde{s}(a)=s(\mu(a))$ then $\tilde{s}\in S$ for every $s\in S$. Let $t\to\alpha_t$ be a one-parameter group of permutations of S such that any one of the following conditions (i) and (ii) (which can be proved to be equivalent) is satisfied:

(i)

$$\begin{cases} \text{(a)}\ L(s)\supset L(D)\Rightarrow L(\alpha_t s)\supset L(\alpha_t D) & \forall t\in\mathbb{R} \quad (s\in S, D\subset S) \\ \text{(b)}\ \bigwedge L(\alpha_t S_1(a))=(\bigwedge L(\alpha_t S_1(a^{\perp})))^{\perp} & \forall t\in\mathbb{R}, \quad a\in L \end{cases}$$

(ii) $\forall a\in L\ \exists b\in L$ such that $\alpha_t S_1(a)=S_1(b)$ and $\alpha_t S_1(a^{\perp})=S_1(b^{\perp})$.

Then there is a one-parameter group $t\to\rho_t$ of convex permutations of S which is implemented (that is, conditions (i)–(v) of Definition 2.9 are fulfilled and hence $t\to\rho_t$ is a dynamical group if it is also weakly continuous), and it is such that

$$\text{(iii)} \qquad \alpha_t S_1(a)=\rho_t S_1(a) \qquad \forall t\in\mathbb{R}, \qquad a\in L$$

Such a one-parameter group $t\to\rho_t$ is defined by

$$\text{(iv)} \quad (\rho_t s)(a)=s(\bigwedge L(\alpha_{-t}S_1(a))) \qquad \forall s\in S, \qquad a\in L$$

Remark 2.7. The one-parameter group $t\to\alpha_t$ of Theorem 2.1 is not necessarily assumed to be made up of convex automorphisms of states, as is shown by the nonlinear example of Zecca (1976).

From the results (iii) of the last theorem we have that $\alpha_t s=\rho_t s\ \forall s\in S_p$, $\forall t\in\mathbb{R}$ since the pure states satisfy the relation $S_1(\bigwedge L(s))=\{s\}$.

Proposition 2.10. Let (L, S) be a *pss* such that S is stable under orthoautomorphisms of L and moreover with the property

$$\text{(i)} \qquad s \in S \Rightarrow s = \sum_1^\infty \gamma_i s_i, \qquad \gamma_i \in [0,1], \sum_1^\infty \gamma_i = 1, \qquad \{s_i\} \subset S_p$$

(σ decomposability of the states of S in terms of the extreme states). Then any weakly continuous one-parameter group $t \to \alpha_t$ of σ-convex permutations of S satisfying condition (i) [or (ii)] or Theorem 2.2 is a dynamical group.

Proof. We have immediately, by taking into account the results of Theorem 2.2 and assumption (i),

$$\alpha_t s = \alpha_t \sum_1^\infty \gamma_i s_i = \sum_1^\infty \gamma_i \alpha_t s_i = \sum_1^\infty \gamma_i \rho_t s_i = \rho_t s \qquad \forall s \in S, \qquad t \in \mathbb{R}$$

the s_i's being pure states.

We note that assumption (i) of Theorem 2.10 implies that if $s \in S_1(a)$ then $S_1(a) \cap S_c \supset \{s_i\}$. Hence L has been implicitly assumed to be an atomic logic as a consequence of Proposition 2.2.

Remark 2.8. We now study Definition 2.9 in the case of the *pss* (M^p, N) of the projections M^p and the normal state N of the W^* algebra M. Let $t \to \alpha_t$ be a dynamical group in the sense of Kadison (1965), that is, a one-parameter group of convex automorphisms of N with the property that the map $t \to (\alpha_t \rho)(A)$ is continuous $\forall A \in M$, $\rho \in N$ [obviously, $t \to \alpha_t$ obeys the conditions (i)–(iv) and (vi) of our Definition 2.9]. By applying the result of Kadison (1965), Theorem 3.3, there exists then a weakly continuous one-parameter group $t \to \mu_t^\alpha$ of Jordan automorphisms of M such that

$$\rho(\mu_t^\alpha(A)) = (\alpha_t \rho)(A) \qquad \forall A \in M, \qquad \rho \in N, \qquad t \in \mathbb{R}$$

Since, as is easily checked, the restriction of a Jordan automorphism to the projections M^p is an orthoautomorphism of the logic M^p, there follows that $t \to \alpha_t$ is implemented (in the sense of Definition 2.9) so that it is a dynamical group also in our sense.

In the special case in which $M = B(H)$, the bounded linear operators on some Hilbert space H, and $N = K(H)$, the statistical operators on H (positive trace class operators with trace 1), we have directly (Mackey, 1963)

$$\alpha_t \rho = U_t \rho U_t^+ \qquad \forall \rho \in K(H), \qquad t \in \mathbb{R}$$

$t \rightarrow U_t$ being a strongly continuous one-parameter group of unitary operators on H. The generator of $t \rightarrow U_t$, which exists by Stone's theorem (Reed and Simon, 1972), can then be interpreted as the Hamiltonian of the system.

3. THE IRREDUCIBLE STANDARD HILBERT MODEL

3.1. Superposition of Statistical Operators. The proposition-state structure can be specialized to the usual Hilbert model as a consequence of an important result that we immediately state and for the proof of which we refer to the book of Maeda and Maeda (1970).

> *Theorem 3.1.* Let L be a complete, orthocomplemented, irreducible atomistic lattice with the covering property and of length $\geqslant 4$. Then there is a division ring K, an involutorial antiautomorphism $\lambda \rightarrow \lambda^*$ of K, a vector space E over K, and a Hermitian form f such that L is orthoisomorphic to the lattice $L_E(E)$ of the E-closed subspace of E. (By setting $M^0 = \{x \in E\colon f(x, y) = 0 \ \forall y \in M\}$, the subspace M of E is said to be E-closed if $M = M^{00}$ holds).

In the theorem the choice of the division ring K remains open. To get the ordinary quantum mechanics we assume

$$K = C \text{ (complex numbers) and } \lambda^* = \bar{\lambda} \text{ (complex conjugation)}$$

We are now in a position to formulate Piron's result.

> *Theorem 3.2.* Suppose that L satisfies, besides the assumption of Theorem 3.1, also the previous complex field assumption. Then a necessary and sufficient condition in order that L be isomorphic to the lattice $L(H)$ of the closed subspaces of a separable complex Hilbert space H is that L be orthomodular and that every family of mutually orthogonal atoms of L be at most countable.

The order relation and the lattice operations in $L(H)$, which will be denoted again with the symbols $\leqslant, \wedge, \vee, \perp$, have the following meaning. If $a, b \in L(H)$, $a \leqslant b$ stands for set inclusion; $a \wedge b$ stands for the set theoretical meet; $a \vee b$ stands for the closed linear span generated by $a \cup b$ in H; and $a^{\perp}$ stands for the Hilbertian orthogonal complement of a in H.

Let now (L, S) be a *pss* for the physical system Σ. We assume through this section that, besides A1–A5, also the following conditions are fulfilled:

A6: The logic L satisfies the conditions of Theorem 3.2 under which it comes out to be orthoisomorphic to a Hilbertian logic $L(H)$ (we denote by π the ortho-isomorphism under consideration).

A7: The set of the states coincides with the set of *all* the generalized probability measures on L, namely, with the set of all maps s: $L \to [0,1]$ such that

$$s\left(\bigvee_i a_i\right) = \sum_{i=1}^{\infty} s(a_i)$$

for every countable family $\{a_i\}$ of mutually orthogonal elements of L.

By setting $m_s(x) = s(\pi^{-1}(x))$, $x \in L(H)$ $(s \in S)$ we have that $\{m_s : s \in S\} = S(H)$, $S(H)$ being the set of all generalized probability measures on the logic $L(H)$. By the Gleason theorem which establishes the existence of a convex isomorphism between $S(H)$ and the set of all the density operators $K(H)$ of H (positive trace class operators on H with trace 1) we have then that for every $m \in S(H)$ there is a $\rho \in K(H)$ such that

$$m(x) = m_\rho(x) = \operatorname{Tr} P^x \rho, \qquad x \in L(H)$$

where P^x is the orthogonal projection on H with range x. By using the spectral decomposition of a density operator

$$\rho = \sum_i \gamma_i P^{a_i}$$

where the a_i's are the eigenspaces of ρ in the range of the latter, one gets

$$S_1(a) = \{m_\rho \in S(H) : P^a \rho = \rho\} \qquad [a \in L(H)]$$

Therefore $(L(H), S(H))$ is indeed a *pss* (Gorini and Zecca, 1975) and the physical system can be equivalently described in terms of it instead of in terms of the original *pss* (L, S).

We have the following formulas (Gorini and Zecca, 1975):

$$\wedge L(m_\rho) = \bigvee_i a_i = [\rho] \text{ (the range in } H \text{ of the operator } \rho)$$

$$\wedge L(D) = \bigvee_{m_\rho \in D} [\rho], \qquad D \subset S(H)$$

(The connection with the case of the W^* algebras of Remark 2.1 is given by the fact that the support of the trace state ρ is the orthogonal projection in H whose range is exactly $[\rho]$.)

By taking into account the spectral decomposition of a density operator we have, moreover, that a state $m_\rho \in S(H)$ *is pure if and only if* there is a vector $\psi \in H$, $\|\psi\| = 1$ such that

$$m_\rho(x) = m_\psi(x) = \mathrm{Tr}\, P^x P^\psi = \|P^x \psi\|^2, \qquad x \in L(H)$$

where P^ψ is the orthogonal projection on the one-dimensional subspace of H generated by ψ.

There is then a bijection between the pure states of $S(H)$ and the rays of H.

It is now possible to give a transparent interpretation of the superposition relation for the density operators of the physical system. Indeed if $m_\rho \in S(H)$, $D \subset S(H)$ we have, by taking into account the previous considerations, that $L(m_\rho) \supset L(D)$ if and only if

$$[\rho] \subset \bigvee_{m_\sigma \in D} [\sigma] \tag{3.1}$$

This means that m_ρ is a superposition of the states of $D \subset S(H)$ if and only if each unit vector ψ_i of the spectral decomposition of $\rho = \Sigma \gamma_i P^{\psi_i}$ in terms of one-dimensional projections (obtained by repeating the eigenvalues γ_i if necessary) is a linear combination of unit vectors obtained by analogously decomposing the density operators σ such that $m_\sigma \in D$. In this sense our superposition relation is an extension to the statistical operators of Dirac's superposition of pure states. In particular if $m_\rho = m_\psi$ is a pure state and $D = \{m_{\psi_1}, m_{\psi_2}\}$ consists of pure states too, from relation (3.1) we have $\psi = c_1\psi_1 + c_2\psi_2$ since $[\psi_1] \vee [\psi_2]$ is the linear span in H of ψ_1 and ψ_2.

By contracting our language, we will refer to the relation (3.1) as the superposition relation for the statistical operators of the system.

3.2. Superposition and Irreversible Dynamics. The object of this section is that of showing that the superposition relation of the statistical operators is compatible with the most general (linear) dynamical evolution of the physical system.

Definition 3.1. A *dynamical map* B for the physical system Σ with hilbert space H is a map of the statistical operators (density operators) $K(H)$ into themselves which is affine, namely, such that

$$B(\alpha\rho + (1-\alpha)\sigma) = \alpha B\rho + (1-\alpha)B\sigma \qquad \forall \rho, \sigma \in K(H), \qquad 0 \leqslant \alpha \leqslant 1$$

(B is not assumed to be onto nor one to one.) To show the main result of the section we need the following proposition.

Proposition 3.1. Let F be a positive linear operator on H such that $F \leqslant \mathbb{1}$ (identity map in H) and $D \subset K(H)$ any family of density operators. Then the following conditions are equivalent:

(i) $F\psi = \psi \quad \forall \psi \in \bigvee_{\sigma \in D} [\sigma]$

(ii) $\mathrm{Tr}\, F\sigma = 1 \quad \forall \sigma \in D$

Proof. (i)⇒(ii). By writing the spectral decomposition of a density operator σ in terms of one-dimensional projections

$$\sigma = \sum_i \alpha_i P^{\psi_i}, \qquad \|\psi_i\| = 1 \ \forall i$$

we have $\psi_i \in [\sigma] \ \forall i$. Hence, by assumption (i), $\mathrm{Tr}\, F\sigma = \Sigma_i \alpha_i \mathrm{Tr}\, FP^{\psi_i} = \Sigma_i \alpha_i \mathrm{Tr}\, P^{\psi_i} = 1 \ \forall \sigma \in D$.

(ii)⇒(i). If $\sigma = \Sigma_i \alpha_i P^{\psi_i}$ we have $\mathrm{Tr}\, F\sigma = 1$ if and only if $\mathrm{Tr}\, FP^{\psi_i} = 1 \ \forall i$, that is if and only if $(\psi_i, F\psi_i) = 1 \ \forall i$. By setting $F\psi_i = \psi_i + \varphi_i$, we have then $(\psi_i, \psi_i) + (\psi_i, \varphi_i) = 1$ so that $(\psi_i, \varphi_i) = 0 \ \forall i$. But $\|F\psi_i\| = 1 \ \forall i$ since

$$1 = |(\psi_i, F\psi_i)| \leqslant \|F\psi_i\| \leqslant \|\psi_i\| = 1 \ \forall i$$

by the Schwartz inequality and the assumptions on F. Hence $(\psi_i, \psi_i) + (\varphi_i, \varphi_i) = 1$ so that $\varphi_i = 0 \ \forall i$. The above reasoning holds for every σ in D. Hence $F\psi = \psi \ \forall \psi \in [\sigma], \sigma \in D$.

If now $\psi \in \bigvee_{\sigma \in D} [\sigma]$ then $\psi = \lim_n x_n$ with $\{x_n\} \subset \Sigma_\sigma [\sigma]$, where Σ denotes the algebraic sum. There follows $F\psi = F \lim x_n = \lim x_n = \psi$.

Proposition 3.2. Any dynamical map B of the physical system preserves the superposition relation of the statistical operators, namely,

$$L(\rho) \supset L(D) \Rightarrow L(B\rho) \supset L(BD) \qquad (\rho \in K(H), D \subset K(H))$$

[We have written $L(\rho)$ instead of $L(m_\rho)$ to avoid heavy notation.]

Proof. The linear extension $\tilde{B}$ of the dynamical map B to the trace class operators $T(H)$ on H is a positive preserving trace linear map of $T(H)$ into itself. The dual map $\tilde{B}^*$ of $\tilde{B}$ is the positive map of the bounded operators $B(H)$ of H into themselves defined by (Schatten, 1960)

$$\mathrm{Tr}(\tilde{B}^* a)\rho = \mathrm{Tr}\, a(\tilde{B}\rho) \qquad \forall \rho \in T(H), \qquad a \in B(H)$$

By choosing $a = \mathbb{1}$ in the last equation and by taking into account that $\tilde{B}$ preserves the trace, we have

$$\tilde{B}^* \mathbb{1} = \mathbb{1}$$

Hence $0 \leqslant \tilde{B}^* P \leqslant \mathbb{1}$ for every projection P of H. Let now $x \in L(BD)$. Then

$$\operatorname{Tr} P^x(\tilde{B}\sigma) = 1 = \operatorname{Tr}(\tilde{B}^* P^x)\sigma \qquad \forall \sigma \in D$$

which is equivalent, by Proposition 3.1 and by the previous considerations, to

$$(\tilde{B}^* P^x)\psi = \psi \qquad \forall \psi \in \bigvee_{\sigma \in D} [\sigma]$$

If $\rho \in K(H)$ is a superposition of the density operators of D, by taking into account (3.1) we have

$$(\tilde{B}^* P^x)\psi = \psi \qquad \forall \psi \in [\rho]$$

and hence, again by Proposition 3.1, $\operatorname{Tr}(\tilde{B}^* P^x)\rho = 1 = \operatorname{Tr}(\tilde{B}\rho)P^x$, that is $x \in L(B)$.

The result of the last proposition enables us to assert that the superposition relation of the statistical operators of the physical system is preserved under the most general (linear) dynamical evolution which the physical system undergoes.

Indeed a dynamical evolution for the physical system for which a statistical interpretation is allowed is described in general in terms of a one-parameter family $t \to B_t$ of dynamical maps, with the following interpretation. If ρ is the state of the system at time $t=0$, then $B_t\rho$ represents the state of the system at time t.

A standard and quite general example of one-parameter family $t \to B_t$ of dynamical maps is that given by the motion of the physical system governed by a homogeneous generalized master equation, which gives a formally exact description of the time evolution of a quantum open system coupled to its surroundings (Haake, 1973; Lanz, Rugiato, and Ramella, 1971; Lugiato, 1976).

We remark that, since B_t has not been assumed to be, in general, a bijection, the superposition of the statistical operators is preserved also under an irreversible dynamical evolution. For instance, $t \to B_t$ ($t \geqslant 0$) could be the one-parameter semigroup of dynamical maps obtained from the solution of a Markovian master equation. These equations are widely used in the phenomenological treatment of open systems and are also a useful tool to approximate, in the weak-coupling limit (Davies, 1974) or in the singular reservoir limit (Hepp and Lieb, 1973) the generalized master equations (see also Frigerio, Gorini, Kossakowski, Sudarshan, and Verri, 1978, and references therein).

A limiting case of the above examples, as well as of Definition 3.1, is that in which B_t is a convex automorphism of $K(H)$ for every $t \in \mathbb{R}$, and the family $t \to B_t$ is a one-parameter group. If the weak continuity is also assumed, then one recovers the unitary time evolution obtained at the end of Remark 2.8.

3.3. Superposition and Tensor Product. We will suppose in this section that the physical system Σ is coupled to the physical system $\tilde{\Sigma}$ and that the proposition-state structures associated with both systems satisfy our axioms A1–A7. Hence Σ and $\tilde{\Sigma}$ can be described in terms of the proposition-state structures $(L(H), K(H))$ and $(L(\tilde{H}), K(\tilde{H}))$, respectively ($H, \tilde{H}$ separable complex hilbert spaces). Then the question arises, what kind of *pss* can be associated to the compound physical system $\Sigma + \tilde{\Sigma}$? According to the standard formulation of quantum mechanics one associates to $\Sigma + \tilde{\Sigma}$ the Hilbert space $H \otimes \tilde{H}$ ($\otimes$ is the usual tensor product), so that, according to our point of view, $\Sigma + \tilde{\Sigma}$ can be described in terms of the *pss* $(L(H \otimes \tilde{H}), K(H \otimes \tilde{H}))$ with the following physical interpretation. A product proposition $a \otimes \tilde{a}$ of $L(H \otimes \tilde{H})$ is a yes–no experiment on $\Sigma + \tilde{\Sigma}$ which consists in testing Σ with a and $\tilde{\Sigma}$ with $\tilde{a}$ and then taking the answer yes when both experiments give the answer yes, and no otherwise. The other elements of $L(H \otimes \tilde{H})$ represent the yes–no experiments on $\Sigma + \tilde{\Sigma}$ which cannot be reduced to the product of a test on Σ with a test on $\tilde{\Sigma}$. We remark that also when Σ and $\tilde{\Sigma}$ are classical physical systems there exist tests of the compound system $\Sigma + \tilde{\Sigma}$ which are not product tests. This is a consequence of the interpretation of the subsets of the phase space of a classical system in terms of yes–no experiments (see the Introduction) and the fact that a subset of the phase space of $\Sigma + \tilde{\Sigma}$ is not in general the (Cartesian) product of a subset of the phase space of Σ with a subset of the phase space of $\tilde{\Sigma}$. Instead a difference is that, while in the quantum case there are atomic propositions of $\Sigma + \tilde{\Sigma}$ which are not the (tensor) product of *atomic* propositions of Σ and $\tilde{\Sigma}$, in the classical (atomic standard) case *every atomic* proposition of $\Sigma + \tilde{\Sigma}$ is the (Cartesian) product of an atomic propositions of Σ with an atomic proposition of $\tilde{\Sigma}$.

Analogous considerations can be given for the states $K(H \otimes \tilde{H})$ of $\Sigma + \tilde{\Sigma}$. A product state $\rho \otimes \tilde{\rho}$ is interpreted as a preparing procedure for $\Sigma + \tilde{\Sigma}$ consisting in preparing Σ according to ρ and $\tilde{\Sigma}$ according to $\tilde{\rho}$. Also here there are states of $\Sigma + \tilde{\Sigma}$ which are not product states, and this holds in particular for the pure states.

According to the interpretation given one expects that the probability factorizes when a product test is performed on $\Sigma + \tilde{\Sigma}$ which has been prepared with the instructions of a product state. This is indeed so, since from the properties of the tensor product there follows

$$\operatorname{Tr} P^{a\otimes\tilde{a}}\rho\otimes\tilde{\rho}=\operatorname{Tr} P^{a}\rho\cdot\operatorname{Tr} P^{\tilde{a}}\tilde{\rho},\quad \rho\otimes\tilde{\rho}\in K(H\otimes\tilde{H}),\qquad a\otimes\tilde{a}\in L(H\otimes\tilde{H})$$

We want now to mention some other properties of the scheme which are connected to the superposition of the statistical operators. To this end we need a simple result.

Lemma 3.1. Let ρ be a statistical operator of Σ and $\tilde{\rho}$ be a statistical operator of $\tilde{\Sigma}$. Then the ranges of the two operators are in the relation

$$[\rho]\otimes[\tilde{\rho}]=[\rho\otimes\tilde{\rho}]$$

Proof. Let P ($\tilde{P}$) be the orthogonal projection on H ($\tilde{H}$) whose range is $[\rho]$ ($[\tilde{\rho}]$). If $\{P_{\psi_i}\}$, ($\{P_{\tilde{\psi}_k}\}$) are the one-dimensional projections of the spectral decomposition of ρ ($\tilde{\rho}$) in the range of the latter, then $P=\oplus_i P_{\psi_i}$, $\tilde{P}=\oplus_k P_{\tilde{\psi}_k}$. It follows that

$$P\otimes\tilde{P}=\bigoplus_{i,k}\left(P_{\psi_i}\otimes P_{\tilde{\psi}_k}\right)=\bigoplus_{i,k}P_{\psi_i\otimes\tilde{\psi}_k}$$

But the last term is exactly the orthogonal projection whose range is $[\rho\otimes\tilde{\rho}]$.

Suppose now $\rho\in K(H)$ is a superposition of the statistical operators of $D\subset K(H)$ and that $\tilde{\rho}\in K(\tilde{H})$ is a superposition of the statistical operators of $\tilde{D}\subset K(\tilde{H})$. By taking into account the relation (3.1), this is equivalent to the assumptions

$$[\rho]\leqslant\bigvee_{\sigma\in D}[\sigma]\quad\text{and}\quad[\tilde{\rho}]\leqslant\bigvee_{\tilde{\sigma}\in\tilde{D}}[\tilde{\sigma}]$$

By the properties of the tensor product and by using Lemma 3.1 we have then

$$[\rho\otimes\tilde{\rho}]\subset\bigvee_{\substack{\sigma\in D\\ \tilde{\sigma}\in\tilde{D}}}[\sigma\otimes\tilde{\sigma}]$$

which, again by formula (3.1), means that the state $\rho\otimes\tilde{\rho}$ is a superposition of the states of $D\otimes\tilde{D}=\{\sigma\otimes\tilde{\sigma}\colon \sigma\in D,\ \tilde{\sigma}\in\tilde{D}\}\subset K(H\otimes\tilde{H})$.

We have thus checked that the superposition is preserved under tensor product.

Finally we want to mention a property of the closure under superposition. From the results of Section 3.1 we have first

$$S_1(a)=\{\rho\in K(H)\colon [\rho]\leqslant a\},\qquad a\in L(H)$$

If now $a \in L(H)$, $\tilde{a} \in L(\tilde{H})$, by using Lemmas 2.1 and 3.1 there follows

$$\wedge L(S_1(a) \otimes S_1(\tilde{a})) = \bigvee_{\substack{[\rho] < a \\ [\tilde{\rho}] < \tilde{a}}} (\wedge L(\rho \otimes \tilde{\rho})$$

$$= \bigvee_{\substack{[\rho] < a \\ [\tilde{\rho}] < \tilde{a}}} [\rho \otimes \tilde{\rho}] = a \otimes \tilde{a}$$

and hence

$$S_1(a \otimes \tilde{a}) = \overline{S_1(a) \otimes S_1(\tilde{a})}$$

as a consequence of the properties of closure under superposition (see Section 2.5).

By setting $a = \mathbb{1}$, $\tilde{a} = \tilde{\mathbb{1}}$ in the last result we have

$$K(H \otimes \tilde{H}) = \overline{K(H) \otimes K(\tilde{H})}$$

We have mentioned above that the coupling of the physical systems produces also states that are not product states. The last equation establishes that such states are exactly all the (quantum) superpositions of the product states.

4. SOME OPEN PROBLEMS

In the previous sections we have proposed a possible axiomatic description of the physical system in terms of logic and states and have studied only those aspects of the theory which are, in our opinion, more related to the superposition of the states. However, to have a complete and satisfactory theory, one has to go on with the program of reproducing and generalizing to the level of a general proposition-state structure the main aspects of the conventional quantum mechanics.

In doing that, however, many problems arise which are still unsolved and which are "not only of interest in their own right, but the solution of which could very likely shed additional light on the basic structure of quantum logics" (Gudder, 1978). For instance, while the characterization of the compatibility of the observables in quantum logics has been solved in Varadarajan (1965) and Gudder (1965), there is open the problem of characterizing those quantum logics for which the observables are determined by their expectations or for which the sum of two arbitrary bounded observables exists. Also the concept of transition probability and the Heisenberg uncertainty principle have not had a complete definition and generalization for quantum logics. (For more details on the status of these problems as well as for information on some other unsolved problems, we refer to Gudder, 1978, and references therein.)

We want here to mention only one other problem which seems to be unavoidable if a theory of quantum measurement is addressed and developed for quantum logics. Suppose we have the proposition-state structures (L, S), $(\tilde{L}, \tilde{S})$ corresponding to the physical systems Σ, $\tilde{\Sigma}$, respectively (for instance Σ could be the physical system under study and $\tilde{\Sigma}$ a measuring apparatus). If one wants to study the compound physical system $\Sigma+\tilde{\Sigma}$ in the context of the quantum logics, one has to associate to $\Sigma+\Sigma$ a proposition-state structure, say $(L\otimes\tilde{L}, S\otimes\tilde{S})$, which should be a (to be defined) well-determined "tensor product" of both (L, S) and $(\tilde{L}, \tilde{S})$ satisfying the appropriate requirements of existence and uniqueness.

Unfortunately, not only has that problem not been solved, but also the related problem of proving the existence of a reasonable "tensor product" of quantum logics has not had a solution, as far as the author knows.

There are only some indications in that direction. Indeed if Σ and $\tilde{\Sigma}$ are classical physical systems with phase space A and $\tilde{A}$, respectively and associated logics $L=\mathcal{P}(A)$, $\tilde{L}=\mathcal{P}(\tilde{A})$, the power set of their phase space, then a logic for the compound system $\Sigma+\tilde{\Sigma}$ can be provided by the power set of the Cartesian product of A and $\tilde{A}$, namely, by $L\otimes\tilde{L}=\mathcal{P}(A\times\tilde{A})$.

If, on the other hand, both Σ and $\tilde{\Sigma}$ are purely quantum physical systems with Hilbertian logic $L=L(H)$ and $\tilde{L}=L(\tilde{H})$, respectively, then a logic for $\Sigma+\tilde{\Sigma}$ can be provided by the tensor product logic, namely, by choosing $L\otimes\tilde{L}=L(H\otimes\tilde{H})$. As shown in Zecca (1978) and Aerts and Daubechies (1978) it is possible to define an intrinsic notion of product for a special class of logics which, besides the Cartesian product and the tensor product, contains also, as a special case, the mixed situation, namely, the one obtained by choosing $L=L(H)$ and $\tilde{L}=\mathcal{P}(\tilde{A})$. The result in this last case is that the compound system $\Sigma+\tilde{\Sigma}$ can be interpreted as a quantum system with continuous superselection rules, an assumption that has been made also in Piron (1976).

Once not only the problem of the coupling of the logics has been solved, but also the related problem of the states has had its solution, there would be, in principle, the possibility of studying the dynamical evolution of the physical system due to the measurement interaction.

Indeed, one could try to define the dynamical evolution of the physical system Σ in interaction with the apparatus $\tilde{\Sigma}$ by restricting to Σ (in a way to be found) a reversible dynamics (which has been provided in Section 2.7) of the compound isolated physical system $\Sigma+\tilde{\Sigma}$.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor P. Caldirola for encouragement in letting the argument of the paper be the subject of a thesis for the "Diploma di Perfezionamento in Fisica Atomica e Nucleare (Milano)." The author is also indebted to G. M. Prosperi for continuous stimulating critical discussions and comments on the subject and to A. Frigerio for invaluable information and contribution in solving some problems connected with the W^*-algebra model.

APPENDIX: DEFINITIONS AND PROPERTIES OF LOGICS

We recall that a *complete lattice* L is a partially ordered set (with partial order relation $\leqslant$) such that for every family A of elements of L there exists in L the least upper bound (denoted $\bigvee A$ and called the join of the elements of A) and the greatest lower bound (denoted $\bigwedge A$ and called the meet of the elements of A) with respect to the order relation $\leqslant$. We denote $\bigvee L = \mathbb{1}$ (one) and $\bigwedge L = \phi$ (zero). An *atom* of L is an element $e \in L$ such that $x \in L$, $x \leqslant e$, implies $x = \phi$ or $x = e$. We denote by $A(L)$ the set of the atoms of L.

The lattice L is said to be *atomic* if for every $a \in L$ there is an $e \in A(L)$ such that $e \leqslant a$; L is said to be *atomistic* if every a of L, $a \neq \phi$, is the join of atoms contained in it.

In the complete lattice L we say that b *covers* a when and $a \leqslant x \leqslant b$ imply $x = a$ or $x = b$. L is said to have the *covering property* if $p \in A(L)$, $a \in L$, and $p \wedge a = \phi$ then $a \vee p$ covers a. An element of the complete lattice L is called a *finite element* when it is either ϕ or the join of a finite number of atoms.

If L has the covering property and a is a finite element of L, then there exists a finite family $\{p_1, \ldots, p_n\}$ of atoms such that $a = \bigvee_i^n p_i$ and $(\bigvee_{k=1}^{i-1} p_k) \wedge p_i = \phi$, $i = 1, \ldots, n$: the number n is called the *dimension* of a. The dimension of $\mathbb{1}$ is called the *length* of L.

An orthocomplementation for the lattice L with ϕ and $\mathbb{1}$ is a bijection $\perp$ from L onto L such that

(i) $x \wedge x^\perp = \phi, \quad x \vee x^\perp = \mathbb{1} \quad \forall x \in L$

(ii) $(x^\perp)^\perp = x \quad \forall x \in L$

(iii) $x \leqslant y \Leftrightarrow y^\perp \leqslant x^\perp \quad x, y \in L$

The element $x \in L$ is said to be orthogonal to $y \in L$ (written $x \perp y$) if $x \leqslant y^\perp$ (equivalently $y \leqslant x^\perp$).

An *orthomodular lattice* is an orthocomplemented lattice with the property

(iv) $x, y \in L, \quad x \leqslant y \Rightarrow y = x \vee (x^\perp \wedge y)$

We call *logic* a complete orthomodular lattice. We say that x commutes with y (xCy) in the logic L if

(v) $x=(x\wedge y)\vee(x\wedge y^{\perp})$

xCy holds if and only if yCx (Maeda and Maeda, 1970, Lemma 36.3). The *center* $C(L)$ of the logic L is the set

(vi) $C(L)=\{z\in L\colon zCy\ \forall y\in L\}$

The center is a distributive sublogic of L, namely, besides the properties of a logic it has also the property

(vii) $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)\ \forall x,y,z\in C(L)$

and the dual relation of (vii) obtained by interchanging $\wedge$ with $\vee$ (Holland, 1963).

The *atoms of the center* will be denoted by $A(C(L))$. If $x\in L$, L a logic, the *central cover* of x ($e(x)$) is the least element of $C(L)$ which contains x.

For every element a of the logic L, the set

(viii) $L[\phi,a]=\{x\in L\colon \phi\leqslant x\leqslant a\}$

is a logic when endowed with the relative orthocomplementation $x\to x'=x^{\perp}\wedge a$ (Maeda and Maeda, 1970, Lemma 29.15).

If L, M are logics, a bijection μ from L onto M with the properties

(ix) $$\begin{cases} a\leqslant b\Leftrightarrow\mu(a)\leqslant\mu(b), & a,b\in L\\ \mu(a^{\perp})=\mu(a)^{\perp} & \forall a\in L\end{cases}$$

(by $\leqslant,\perp$ we have denoted the partial ordering and the orthocomplementation both in L and in M) is said to be an *orthoisomorphism* of L onto M (*orthoautomorphism* if $L\equiv M$). An orthoisomorphism preserves also the join and the meet (Jauch, 1968, Lemma 9.4.1) and hence the whole structure of a logic.

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